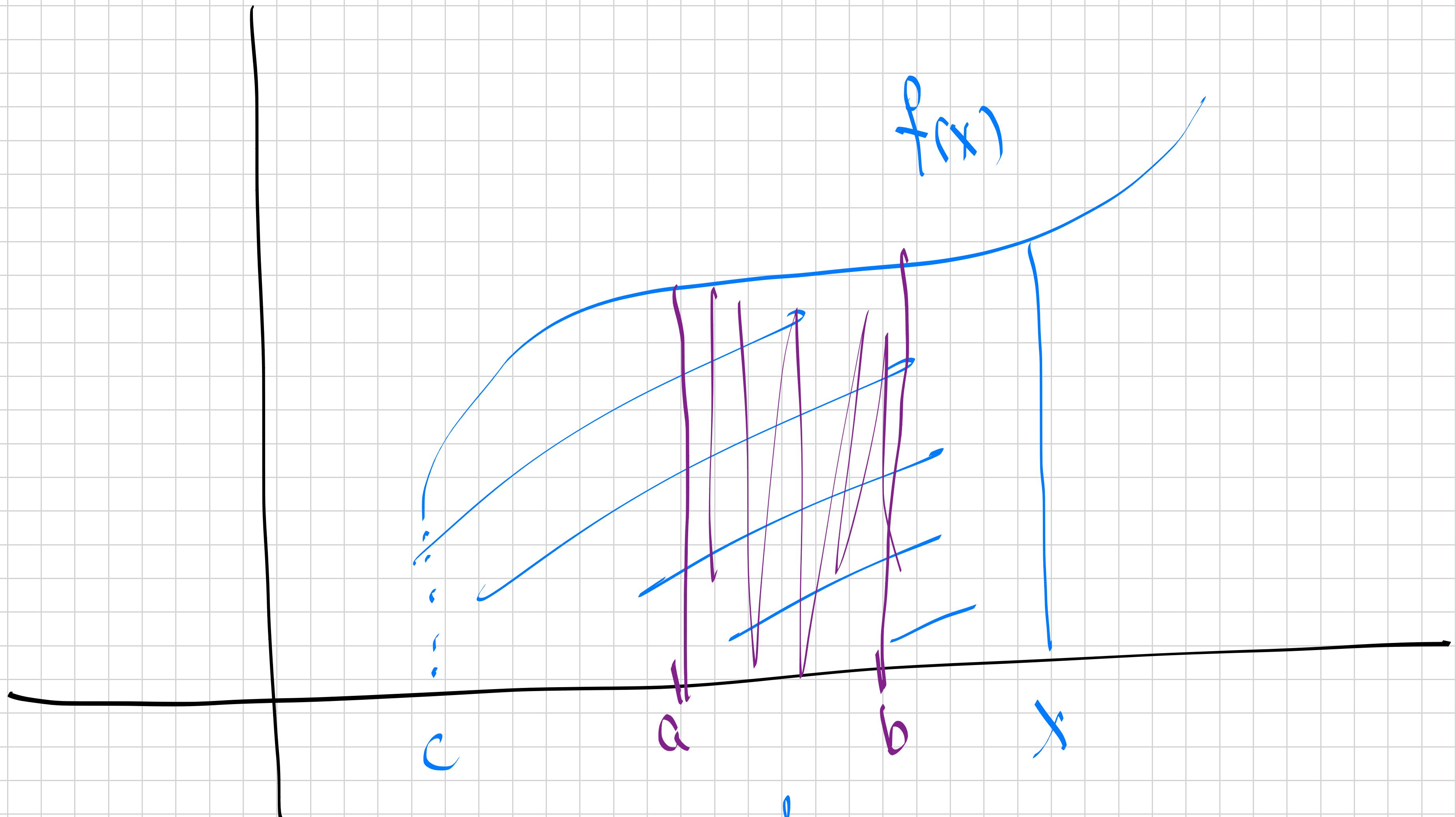
Lesson 34
Working with integrals (5.4)
Substitution (5.5)

Today: * Definite integrals of even and odd functions
* Average value of function
* Substitution Rule
* Examples using Substitution Rule

Announcements: No class on Monday (11/24)
No Recitation on Tuesday (11/25)
Final Exam: Monday (12/15), 1pm - 3pm

Review: (Fundamental Theorem of Calculus)

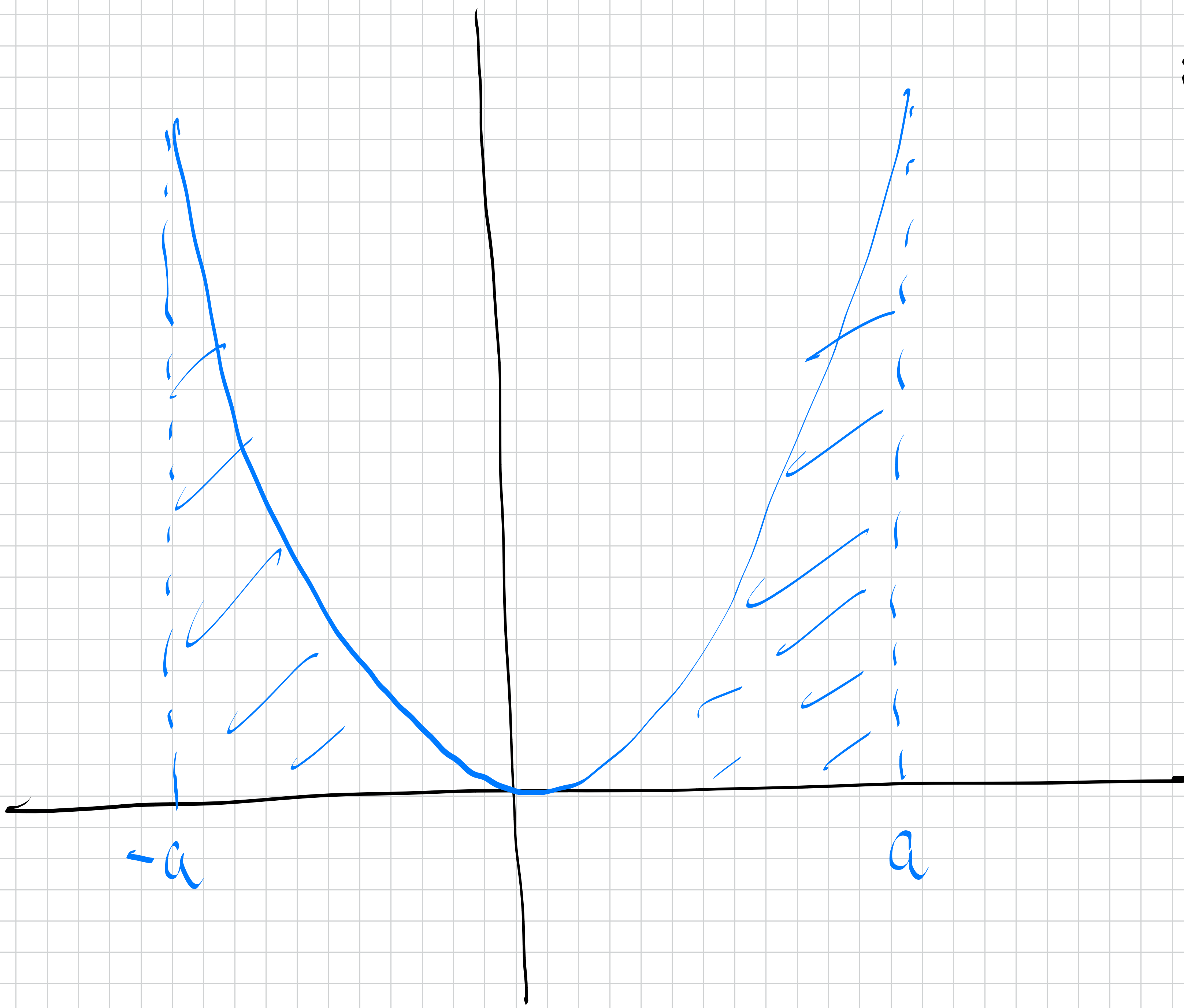
$$A(x) = \int_c^x f(t)\, dt$$



I. $$A'(x) = \frac{d}{dx}\left(\int_c^x f(t)\, dt\right) = f(x)$$ $\left.\right\}$ A(x) is Anti derivative of f(x)

II. $$\int_a^b f(x)\, dx = A(b) - A(a)$$

Definite integral of even functions



* y-axis symmetry
* $f(-x) = f(x)$

  eg: $|x|$, x^2, $\cos x$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

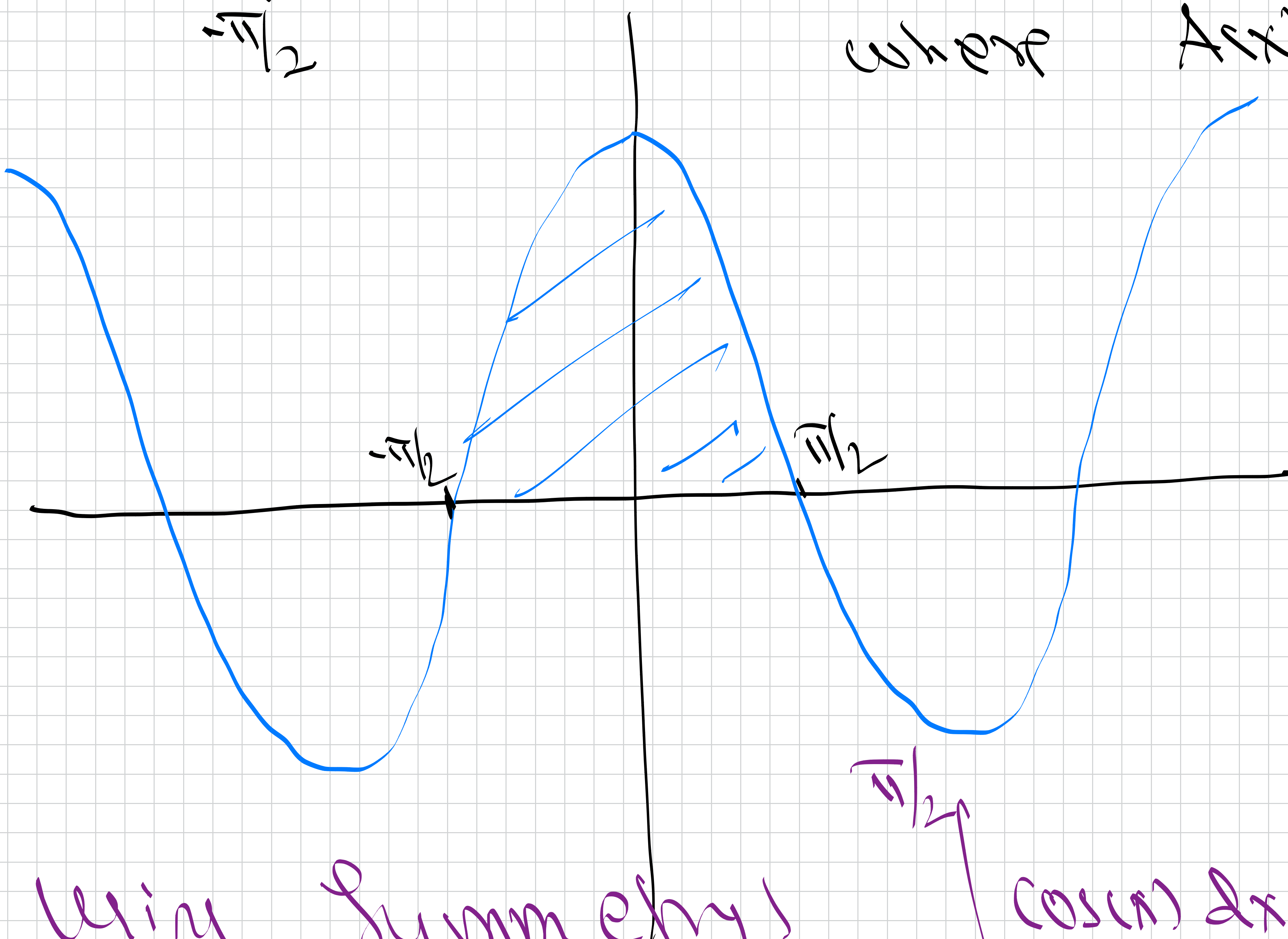
$$= 2\int_{-a}^{0} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$

eg: $\int_{-\pi/2}^{\pi/2} \cos x \, dx = A(\pi/2) - A(-\pi/2)$

where $A(x)$ is antiderivative of $\cos x$

$= \sin x$



$$\int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)$$

$$= 1 - (-1)$$

$$= 2$$

Using symmetry:

$$\int_{-\pi/2}^{\pi/2} \cos(x) \, dx = 2 \int_{0}^{\pi/2} \cos x \, dx = 2\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right]$$

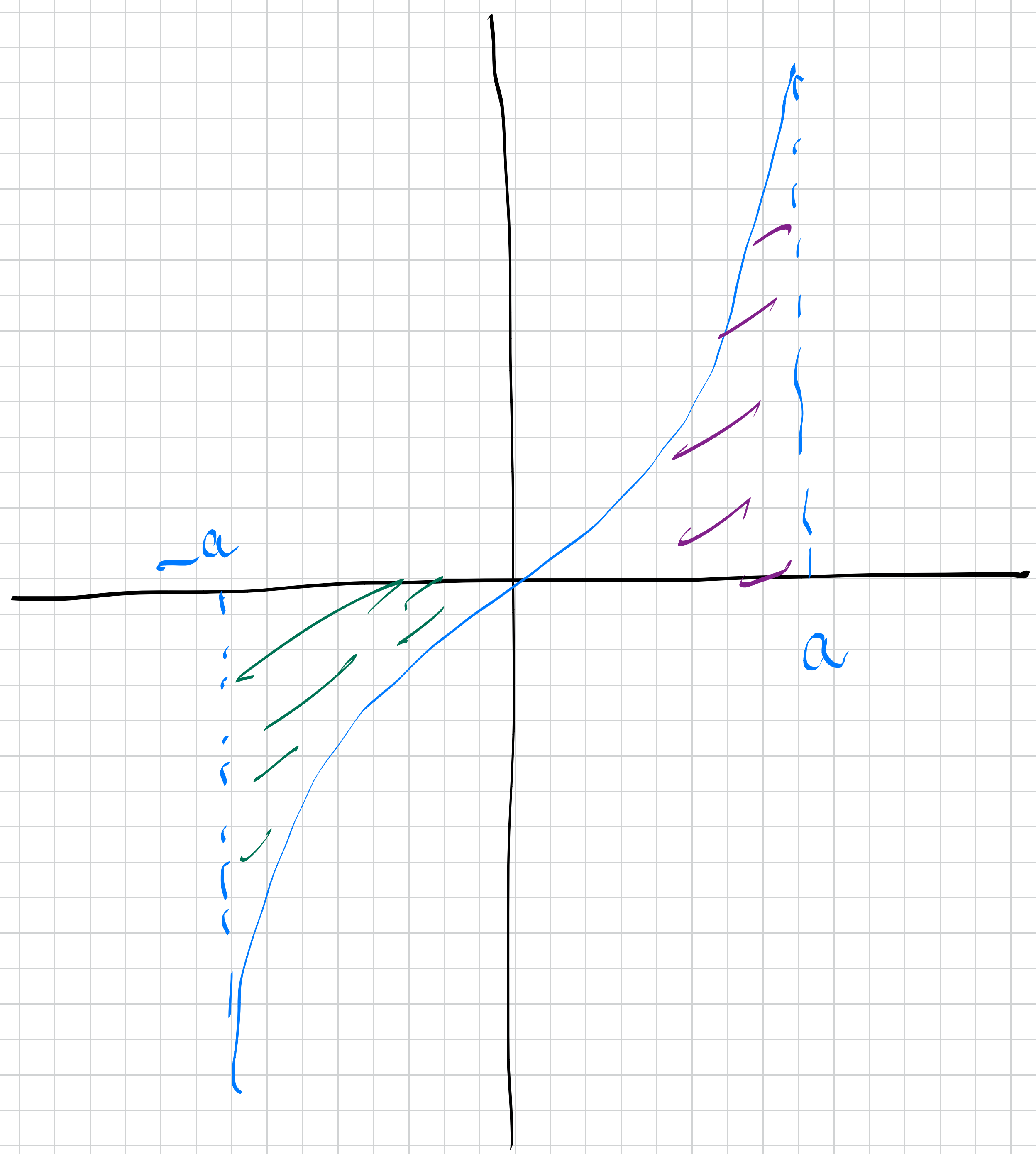
$$= 2$$

Definite integral of odd functions

Origin symmetry

$f(-x) = -f(x)$

Eg: $\sin x$, x^3, x^9 . . .



$$\int_{-a}^{a} f(x)\,dx = \text{Area Above} - \text{Area below}$$

$= 0$.

eg)

$$\int_{-\pi/4}^{\pi/4} \tan^3 x + \sin x - x^3 \, dx$$

$f(x) = \tan^3 x + \sin x - x^3$

$f(-x) = (\tan(-x))^3 + \sin(-x) - (-x)^3$

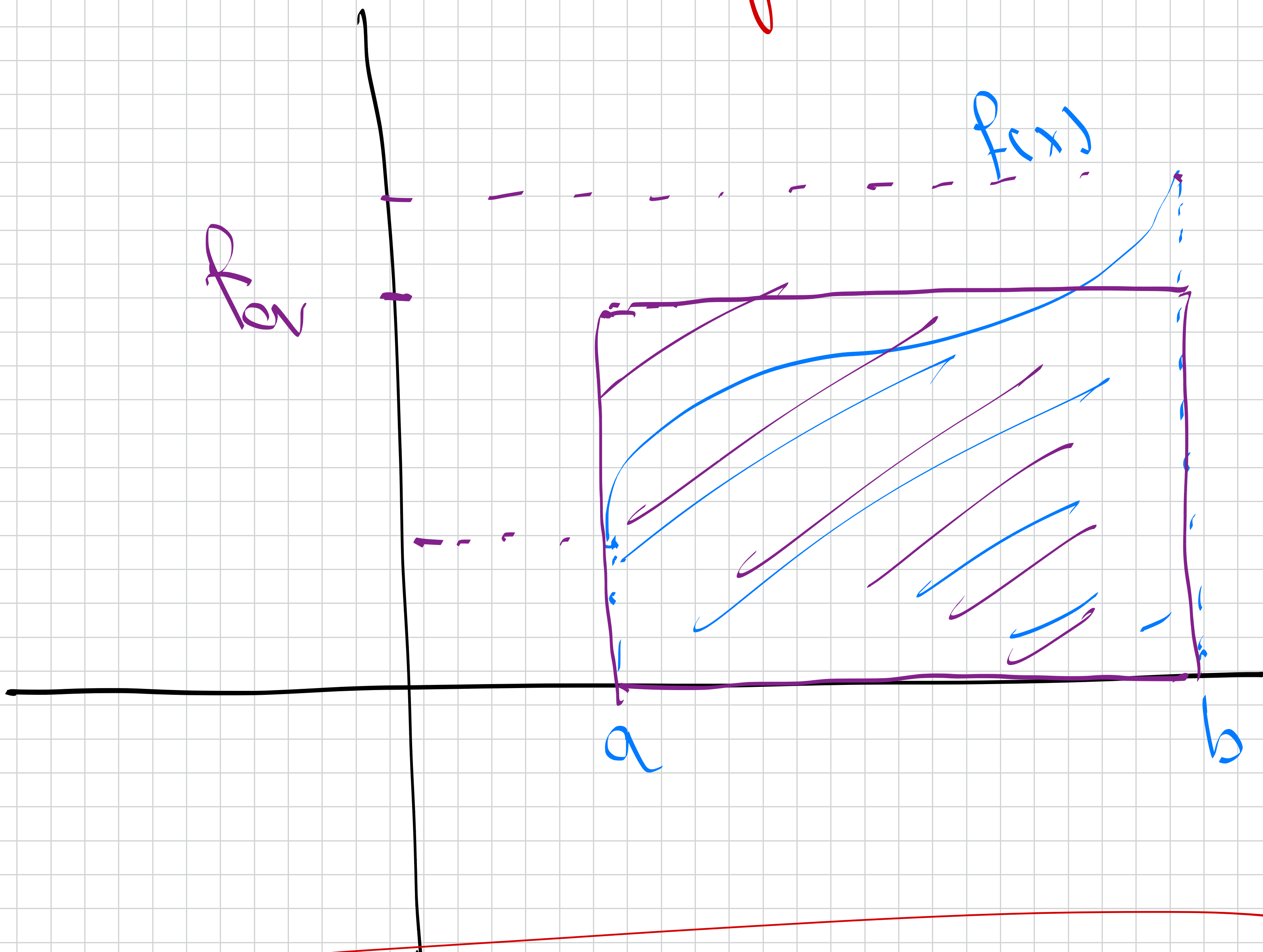
$= (-\tan x)^3 - \sin x + x^3$

$= - f(x)$

$f(x)$ is odd

$$\int_{-\pi/4}^{\pi/4} \tan^3 x + \sin x - x^3 \, dx = 0 \, ,$$

Average value of a Function



$$\int_a^b f(x)\,dx$$

f_{av} is the value of function such that
Area of Rectangle = Area under curve

$$\boxed{\underset{\text{on } [a,b]}{f_{av}} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

eg: What is the average value of $f(x) = (x-1)^2$ on $[1,4]$

$$f_{av.} \text{ on } [1,4] = \frac{1}{4-1}\int_1^4 (x-1)^2\,dx$$

$$= \frac{1}{3}\int_1^4 (x-1)^2\,dx = \frac{1}{3}\int_1^4 x^2-2x+1\,dx$$

$$= \frac{1}{3}\left[\left(\frac{x^3}{3}-x^2+x\right)\Big|_1^4\right]$$

$$= \frac{1}{3}\left[\left(\frac{64}{3}-16+4\right)-\left(\frac{1}{3}\cancel{-1+1}\right)\right]$$

$$= \frac{1}{3}\cdot 9 = 3.$$

Substitution Rule to compute antiderivatives.

Undoing Chain Rule

Recall Chain Rule!

$$\frac{d}{dx} f(\overset{u}{g(x)}) = \frac{d}{dx} f(u)$$

$$= \frac{d}{du} f(u) \; \frac{du}{dx}$$

$$= f'(g(x)) \cdot g'(x)$$

Antiderivative of $\quad f'(g(x)) \cdot g'(x) = f(g(x))$.

Antiderivative of $f'(g(x)) \cdot g'(x)$

$$\int f'(g(x))\, g'(x)\, dx$$

$u = g(x)$

$\frac{du}{dx} = g'(x) \rightsquigarrow du = g'(x)\, dx$

$$\int f'(u)\, du$$

$= f(u) + C$

$= f(g(x))$

$$\int_{x=1}^{x=4} (x-1)^2 \, dx$$

$u = x-1$

$\frac{du}{dx} = 1 \rightsquigarrow du = dx$

$x=1 \rightsquigarrow u = x-1 = 0$, $x=4 \rightsquigarrow u = x-1 = 3$

$$\int_{u=0}^{u=3} u^2 \, du = \frac{u^3}{3}\Big|_0^3 = \frac{3^3}{3} - \frac{0^3}{3}$$

$$= 9 .$$

Evaluating $\int 10\,(x^2+3x+5)^9\,(2x+3)\,dx$

$u = x^2+3x+5$

$\frac{dy}{dx} = 2x+3 \rightsquigarrow du = 2x+3\,dx$

$$= \int 10\,u^9\,du$$

$$= 10\cdot\frac{u^{9+1}}{9+1} + C = 10\,\frac{u^{10}}{10} + C = u^{10} + C$$

$$= (x^2+3x+5)^{10} + C$$

derivative using chain Rule

eg: Evaluate $\displaystyle\int \frac{\sin(\sqrt{x})}{2\sqrt{x}}\, dx$

$u = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$

$$= \int \sin(u)\, du$$

$$= -\cos(u) + C$$

$$= -\cos(\sqrt{x}) + C.$$

$$\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin(\sqrt{x})}{2\sqrt{x}}\,dx$$

$u = \sqrt{x}$

$du = 2\sqrt{x}\,dx$

$$\int_{\frac{\pi^2}{4}}^{\pi^2} \sin u\,du = -\cos(u)\Big|_{\frac{\pi^2}{4}}^{\pi^2}$$

$$= -\cos(\pi^2) + \cos(\pi^2/4)$$

We need to change the bounds to u aswell when you have definite integral

Correct method!

$$\int_{\pi^2/4}^{\pi^2} \frac{\sin(\sqrt{x})}{2\sqrt{x}}\,dx$$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$x = \frac{\pi^2}{4}$ ws $u = \pi/2$

$x = \pi^2 \rightsquigarrow u = \pi$

$A(x) = -\cos(\sqrt{x})$ is antiderivative

$$\int_{\pi^2/4}^{\pi^2} \frac{\sin(\sqrt{x})}{2\sqrt{x})} = A(\pi^2) - A\left(\frac{\pi^2}{4}\right)$$

$$= -\cos(\pi) + \cos\left(\frac{\pi}{2}\right)$$

$$= 1$$

$$= \int_{\pi/2}^{\pi} \sin(u)\,du$$

$$= -\cos(u)\Big|_{\pi/2}^{\pi}$$

$$= -\cos\pi + \cos\pi/2$$

$$= 1$$

eg:

$$\int_1^e \frac{\ln x}{x}\,dx$$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x} \rightsquigarrow du = \frac{1}{x}\,dx$

$x = 1 \rightsquigarrow u = \ln 1 = 0$

$x = e \rightsquigarrow u = \ln e = 1$

$$\int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2} - 0$$

$$= \frac{1}{2}.$$